also can be written as

$$R_N^{1/2}C_f = 2a(mx/\Lambda_m)^{1/2}U/U_\infty$$

and both functions u/U_∞ and $R_N^{1/2}C_f$ can be evaluated.

For the special case of a flat plate $dP/dx = 0$, $U/U_\infty = 1 - mx$, and the relations take a simpler form. Results are plotted in Figs. 1 and 2, and it can be seen that agreement with Rossow[4] is quite good for $mx < 0.2$. Further, Rossow points out that his solutions are acceptable only for $mx < 0.2$. For $mx > 0.2$ the integral solution shows the correct asymptotic behavior.

Constant Conductivity $E = 0$

The case of zero electric field may be solved in a similar manner. Since the flow outside the boundary layer is not affected by the magnetic field in this case, the pressure distribution is independent of the magnetic field. The result is the same momentum integral equation developed for the previous case. The difference lies in the fact that dU/dx is now independent of mx, which was not the case previously. For the flat plate, Λ reduces to zero.

These results are plotted in Fig. 1 and are seen to be in agreement with NACA Report 1358.[4] The limitation to small mx again is not present in the integral solution.

Variable Conductivity $E = -U_\infty B_0$

The assumption of a variable conductivity is, in many cases, more realistic. Kantrowitz[6] found that, for high Mach numbers of order 15, $\sigma = \sigma_0(U - u)/U$, where $\sigma_0 = \sigma|_{y=0}$. This assumption used by Rossow[4] therefore is used here, and the following equations are the result:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + mU_\infty(U - u)\frac{u}{U} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} + mU_\infty\frac{\partial u}{\partial y}\left(1 - 2\frac{u}{U}\right) = \nu\frac{\partial^3 u}{\partial y^3}$$

Boundary conditions lead again to a set of equations for a, b, c, d, and e that yield the equations for δ^*/δ and θ/δ in terms of Λ and Λ_m. With these relations, the momentum integral equation may be solved for Λ and Λ_m. For a flat plate, $\Lambda = 0$ and $U = U_\infty$. Figure 1 again shows proper agreement with the exact solution.[4]

One of the principal advantages of the von Karman-Pohlhausen method is its ability to solve the boundary-layer equations once and for all in terms of parameters dependent only upon the shape of the two-dimensional body. The shape factors Λ are then known functions evaluated from the potential flow solution. Similarly, in the MHD case, the solution may be found once and for all in terms of Λ and the magnetic-shape factors Λ_m. With the introduction of the additional compatibility condition, the von Karman-Pohlhausen method often can be extended to other cases in which it previously failed. The MHD boundary-layer example selected is but one such case.

References

[1] McCune, J. E. and Sears, W. R., "On the concepts of moving electric and magnetic fields in magnetohydrodynamics," J. Aerospace Sci. 26, 674–675 (1959).

[2] Schlichting, H., *Boundary Layer Theory* (McGraw-Hill Book Co., Inc., New York, 1960), 4th ed., p. 135.

[3] Pai, S.-L., *Magnetogasdynamics and Plasma Dynamics* (Julius Springer-Verlag, Vienna, 1962), pp. 65–67.

[4] Rossow, V. J., "On flow of electrically conducting fluids over a flat plate in the presence of a transverse magnetic field," NACA Rept. 1358 (1958).

[5] Moffatt, W. C., "Analysis of MHD channel entrance flows using the momentum integral method," AIAA J. 2, 1495–1497 (1964).

[6] Kantrowitz, A., "A survey of physical phenomena occurring in flight at extreme speeds," *Proceedings of the Conference on High-Speed Aeronautics* (Polytechnic Institute of Brooklyn, New York, January 1955), p. 339.

Laminar MHD Channel Entrance Flows

C. W. Tan*
Grumman Aircraft Engineering Corporation, Bethpage, N. Y.

SIMPLE parabolic velocity distribution has been used[1, 2] in applying the integral methods to solutions for magnetohydrodynamics (MHD) channel flows, but it lacks the accuracy of the solution more closely related to the one predicted by Hartmann.[3] Shohet et al.[4] and Dix[5] numerically solved the entrance-region problem by transforming the complete set of governing equations into a set of finite-difference equations. Moffatt,[3] employing the Hartmann-like velocity distribution, presented numerical solutions for pressure distributions but failed to obtain velocity fields, boundary-layer development, and friction factors. All of the forementioned authors, except those of Refs. 1 and 2, failed to obtain closed-form analytical solutions, which this author proposes to obtain with the appropriate Hartmann-like velocity distributions.

The Hartmann channel is considered here. Laminar, two-dimensional, steady, incompressible, low magnetic Reynolds-number flow is assumed, with magnetic and electric fields mutually normal to the flow direction. Hall effects are assumed to be small and thus are neglected. All of the electrical, mechanical, and thermal properties of the moving fluid or of the stationary walls will be assumed constant. Variation of the freestream velocity in the flow direction is allowed inasmuch as it satisfies the over-all conservation of mass for the constant-area channel.

With the usual boundary-layer assumptions, the momentum integral equation in the flow direction (x direction), taking into account the applied transverse magnetic-field intensity B_0, can be written as

$$\tau_\omega = \rho\,(d/dx)\,(U_\infty^2\theta) + U_\infty\,\rho\,(dU_\infty/dx) + \sigma B_0^2\delta^* \quad (1)$$

where θ and δ^* are, respectively, the momentum and the displacement thicknesses, τ_ω is the shear stress at the wall, U_∞ is the freestream velocity, and ρ and σ are the density and the electrical conductivity of the fluid.

Assuming the Hartmann-like velocity distributions, which take the form

$$U/U_\infty = \{\cosh M - \cosh M[1 - (y/\delta)]\}/(\cosh M - 1) \quad (2)$$

where $M \equiv B_0 a(\sigma/\eta)^{1/2}$ is the Hartmann number, η is the viscosity of the fluid, and δ is the boundary-layer thickness measured in the y direction, which is the direction normal to the walls, one obtains

$$\delta^*/\delta = (\sinh M/M - 1)/(\cosh M - 1) \quad (3)$$

$$\theta/\delta = [(1 + \tfrac{1}{2}\cosh M)(\sinh M/M - 2) + \tfrac{3}{2}]/(\cosh M - 1)^2 \quad (4)$$

and

$$\tau_\omega = \eta(U_\infty/\delta)[M\sinh M/(\cosh M - 1)] \quad (5)$$

Satisfying the over-all conservation of mass for the constant-area channel flow, one obtains for the dimensionless freestream velocity (i.e., ratio of freestream velocity to mean velocity)

$$U_\infty' = 1/\{1 + [(1 - \sinh M/M)/(\cosh M - 1)]\delta'\} \quad (6)$$

Received October 14, 1964; revision received March 30, 1965.

* Research Scientist; now Assistant Professor, School of Engineering.

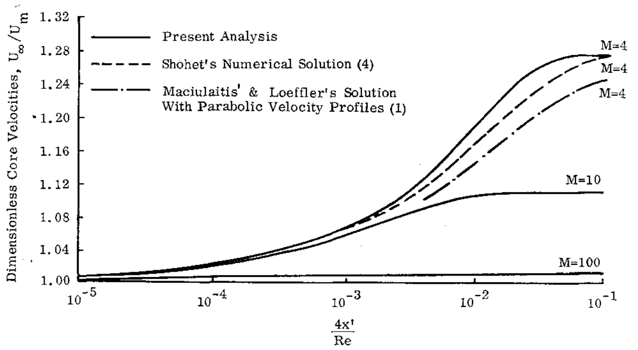

Fig. 1 Development of dimensionless core velocities.

δ' is the ratio of the boundary-layer thickness to the half-height of the channel a.

Substituting Eq. (6) and its derivative and the results of Eqs. (3–5) into Eq. (1), defining $x' \equiv x/a$ and $Re = 4a\rho Um/\eta$, and upon integration by partial fractions, one obtains

$$\frac{(1 - K_6\delta')^{A/K_6}}{(1 + K_6\delta')^{B/K_6}\,(1 + K_7\delta')^{C/K_7}} = \exp\left[-\left(4K_3\frac{x'}{Re} - \frac{D\delta'}{1 + K_7\delta'}\right)\right] \tag{7}$$

where

$$A = \frac{\left(\frac{K_5}{K_6}\right)^2\left\{2\left(\frac{K_7}{K_6}\right)^2 - \left(\frac{K_7}{K_6}\right)\left[1 + \left(\frac{K_7}{K_6}\right)^2\right]\right\} + \left(\frac{K_2}{K_6}\right)\left(\frac{K_7}{K_6}\right)\left[\frac{K_7}{K_6} - \left(\frac{K_7}{K_6}\right)^2 - 1\right]}{2\left(\frac{K_7}{K_6}\right)\left\{2\left(\frac{K_7}{K_6}\right)^2 - \left[1 + \left(\frac{K_7}{K_6}\right)^4\right]\right\}} \tag{8}$$

$$B = \frac{\left(\frac{K_2}{K_6}\right)\left(\frac{K_7}{K_6}\right) - \left\{2\left(\frac{K_7}{K_6}\right)^2 + \left(\frac{K_7}{K_6}\right)\left[1 + \left(\frac{K_7}{K_6}\right)^2\right]\right\}A}{2\left(\frac{K_7}{K_6}\right)^2 - \left(\frac{K_7}{K_6}\right)\left[1 + \left(\frac{K_7}{K_6}\right)^2\right]} \tag{9}$$

$$C = \frac{4\left(\frac{K_7}{K_6}\right)^3 A - \left(\frac{K_2}{K_6}\right)\left(\frac{K_7}{K_6}\right)^2}{2\left(\frac{K_7}{K_6}\right)^2 - \left(\frac{K_7}{K_6}\right)\left[1 + \left(\frac{K_7}{K_6}\right)^2\right]} \tag{10}$$

$$D = \frac{2\left(\frac{K_7}{K_6}\right)^3\left[1 - \left(\frac{K_7}{K_6}\right)^2\right]A - \left(\frac{K_2}{K_6}\right)\left(\frac{K_7}{K_6}\right)^2\left[1 - \left(\frac{K_7}{K_6}\right)\right]}{2\left(\frac{K_7}{K_6}\right)^2 - \left(\frac{K_7}{K_6}\right)\left[1 + \left(\frac{K_7}{K_6}\right)^2\right]} - \left(\frac{K_5}{K_6}\right)^2 \tag{11}$$

and

$$\left.\begin{aligned}
K_1 &= \frac{\sinh M/M - 1}{\cosh M - 1} \\
K_2 &= \frac{(1 + \frac{1}{2}\cosh M)(\sinh M/M - 2) + \frac{3}{2}}{(\cosh M - 1)^2} \\
K_3 &= \frac{M\sinh M}{\cosh M - 1} \qquad K_4 = \frac{\cosh M - \sinh M/M}{\cosh M - 1} \\
K_5 &= [(1 - K_4)(K_1 + K_2)]^{1/2} \\
K_6 &= M(K_1/K_3)^{1/2} \qquad K_7 = K_4 - 1
\end{aligned}\right\} \tag{12}$$

It is appropriate to point out at this point that a trivial solution will be obtained for nonmagnetically fully developed flow ($\delta' = 1$) at the channel entrance (in that the flow remains fully developed all along the channel, as warranted by the assumed velocity distribution that is to force the flow to that of Hartmann flow). The dimensionless friction factor then takes the form

$$f/f_{FD} = K_4/\delta'[1 + K_7\delta'] \tag{13}$$

where f_{FD} is the friction factor when fully developed flow is established at a distance far away from the entrance.

Investigation into the behavior of Eq. (7) shows that an increase in Hartmann number decreases the development length, and that the asymptotic value for δ' for all of the Hartmann numbers at large development length is unity; this behavior of Eq. (7) is in contrast to the flow in which parabolic velocity distributions are assumed, where the asymptotic value takes on different values for different Hartmann numbers.[1] A closer agreement, however, is observed if the comparison is made on the basis of displacement thicknesses rather than on the basis of viscous boundary-layer thicknesses between the two profiles mentioned previously.

A comparison of the development of freestream velocities with Shohet's[4] and Maciulaitis'[1] methods is shown in Fig. 1. Less than 1% discrepancy between Shohet's method and present analysis is observed, whereas Maciulaitis' analysis is about 3% lower than the present analysis, as fully developed flow is achieved. Better agreements are expected at higher Hartmann numbers.

The fully developed friction factor f_{FD} is plotted against $32M/Re$, as shown in Fig. 2. In contrast to turbulent flow where the fully developed friction factor is a function not only of M/Re but also of Re itself, it is worth noting that in steady laminar flow it is a function not only of M/Re but also of M itself. When $M > 100$, however, the friction factor depends upon the ratio M/Re alone. Comparison with Hartmann's and Lazarus' data[6] shows good agreement. Maciulaitis' and Loeffler's data[1] for turbulent flows also are plotted.

It is worth noting also that the dimensionless friction factor f/f_{FD} is a function not only of M^2x'/Re but also of M itself. Although no existing data are available for comparison for various Hartmann numbers, Dix's numerical

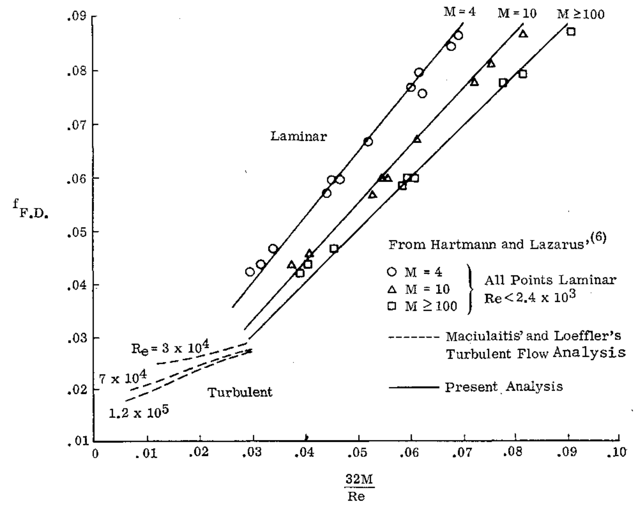

Fig. 2 Variation of friction factor in fully developed flow.

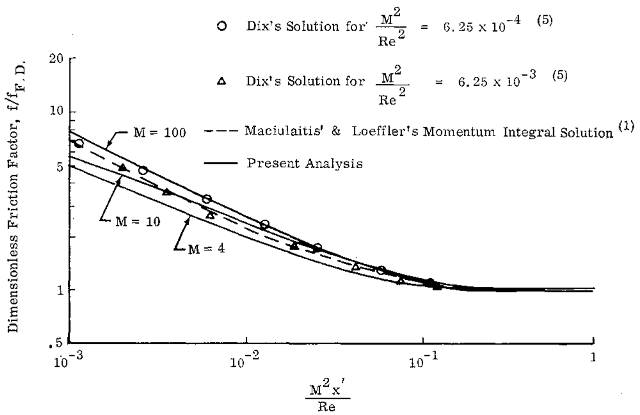

Fig. 3 Comparison with theories of Maciulaitis, Loeffler, and Dix.

data[5] seem to imply that f/f_{FD} increases with increasing Hartmann number, as is observed in the present analysis. Maciulaitis' and Loeffler's integral solution also is plotted for comparison, as shown in Fig. 3.

In view of the foregoing comparisons, it is felt that agreement is good between present investigation and existing data, and that this analysis is an improvement over the existing theories in the literature on MHD channel flows.

References

[1] Maciulaitis, A., and Loeffler, A. L., Jr., "A theoretical investigation of MHD channel entrance flows," Grumman Research Dept. Rept. RE-187J (October 1964); also AIAA J. **2**, 2100–2103 (1964).

[2] Moffatt, W. C., "Boundary layer effects in magnetohydrodynamic flows," Massachussetts Institute of Technology Magnetogasdynamics Lab. Rept. 61-4 (May 1961).

[3] Moffatt, W. C., "Analysis of MHD channel entrance flows using the momentum integral method," AIAA J. **2**, 1495–1497 (1964).

[4] Shohet, J. L., Osterle, J. F., and Young, F. J., "Velocity and temperature profiles for laminar magnetohydrodynamic flow in the entrance region of a plane channel," Phys. Fluids **5**, 545–549 (1962).

[5] Dix, D. M., "The magnetohydrodynamic flow past a nonconducting flat plate in the presence of a transverse magnetic field," Ph.D. Thesis, Massachusetts Institute of Technology; also Massachusetts Institute of Technology TR-397 (1961).

[6] Hartmann, J. and Lazarius, F., "Hg-Dynamics II," Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **15**, 7 (1937).